Alberta-Thy-14-00

Non-Minimally Coupled Massive Scalar Field in a 2D Black Hole: Exactly Solvable Model

V. Frolov[(a)*] and A. Zelnikov[(a,b)†]
[(a)] *Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, AB, Canada T6G 2J1*
[(a)] *P.N. Lebedev Physics Institute, Leninsky pr. 53, Moscow 117924, Russia*

We study a nonminimal massive scalar field in a 2-dimensional black hole spacetime. We consider the black hole which is the solution of the 2d dilaton gravity derived from string-theoretical models. We found an explicit solution in a closed form for all modes and the Green function of the scalar field with an arbitrary mass and a nonminimal coupling to the curvature. Greybody factors, the Hawking radiation, and $\langle\varphi^2\rangle^{\mathrm{ren}}$ are calculated explicitly for this exactly solvable model.

PACS number(s): 03.70.+k, 04.62.+v, 04.70.Dy, 11.10.-z

*Electronic address: frolov@phys.ualberta.ca
†Electronic address: zelnikov@phys.ualberta.ca

I. INTRODUCTION

Study of quantum effects in a black hole spacetime is a very complicated problem. So, an exactly solvable model which exhibits the same qualitative properties as the real black hole could be extremely useful for understanding quantum physics in the real black hole background. First simplification frequently used in the literature is the restriction of the metric and matter fields to have spherical symmetry. The s-wave sector contains almost all features of black holes that makes them such nontrivial objects. Even quantitatively for the Schwarzschild black hole about 90% of Hawking radiation of scalar field is in s-mode. In spite of an apparent simplicity of the s-wave sector, the problem of field quantization on a generic black hole background still remains quite involved. There were developed many powerful methods to approach the problem [1–7]. It would be nice to have a good test for their accuracy and limitations of applicability.

The 2d gravity coupled to a dilaton ϕ field with action

$$S_g = \frac{1}{2\pi}\int d^2x\sqrt{g}\mathrm{e}^{-2\phi}\left[R + 4(\nabla\phi)^2 + 4\lambda^2\right] \tag{I.1}$$

has black hole solutions with the desired properties. First of all, the classical solutions are known in explicit closed form. This action arises in a low-energy asymptotic of string theory models [8–14] and in certain models with a scalar matter [15–17] (For the review of more general dilaton gravity models see [18] and references therein) The corresponding black hole solution has similar properties to those of the (r,t) sector of the Schwarzschild black hole and the Carter-Penrose conformal diagram is also similar. Quantum fields propagating on this background are very similar to the Schwarzschild case but appear to be much simpler to deal with. Most of the papers on this subject consider conformal matter on the 2d black hole spacetimes.

In this paper we address to the problem of quantization of non-conformal fields. This problem is more complicated but much more interesting since nonconformal fields interact with the curvature and feel the potential barrier, which also plays an important role in black hole physics. In this case it is important to know greybody factors to study, e.g., the Hawking radiation and vacuum polarization effects. For the minimally coupled scalar fields in 2d there is no potential barrier and greybody factors are trivial. Qualitative features of the potential barrier of the Schwarzschild black hole are very close to that of the string inspired 2d gravity model we consider here.

II. MODEL

Our purpose is to study a quantum scalar massive field in a spacetime of the 2-dimensional black hole. The most general static 2-dimensional metric can be written in the form

$$dS^2 = -fdT^2 + \frac{dr^2}{f}\,, \tag{II.1}$$

For the theory (I.1) the function f and the corresponding solution for the dilaton field ϕ are

$$f = 1 - e^{-\frac{r}{r_0}}\,, \qquad \phi = \frac{r}{r_0}\,. \tag{II.2}$$

This solution of the dilaton gravity action (I.1) describes the string-theoretic black hole [8,9] . Like the Schwarzschild black hole it is asymptotically flat at $r = \infty$. The metric contains only one parameter r_0, which determines the position of the horizon. The surface gravity κ and the scalar curvature R are

$$\kappa = \frac{1}{2}\frac{df}{dr}|_{r=r_0} = \frac{1}{2r_0}\,, \qquad R = -\frac{d^2 f}{dr^2} = \frac{1}{r_0^2} e^{-r/r_0}\,. \tag{II.3}$$

It would be convenient to use the dimensionless form of (II.1)

$$ds^2 = r_0^{-2}\, dS^2 = -f dt^2 + \frac{dx^2}{f}\,, \tag{II.4}$$

where $t = T/r_0$, $x = r/r_0$, and

$$f = 1 - e^{-x}\,. \tag{II.5}$$

The dimensionless surface gravity, $\tilde{\kappa}$, and curvature, $\tilde{R}$, are

$$\tilde{\kappa} = \frac{1}{2}\,, \qquad \tilde{R} = e^{-x}\,. \tag{II.6}$$

By introducing a new variable

$$z = 1 - e^{-x}\,, \tag{II.7}$$

the metric can be written in an algebraic form

$$ds^2 = -z\, dt^2 + \frac{dz^2}{z\,(1-z)^2}\,. \tag{II.8}$$

We shall also use another form of the metric

$$ds^2 = -z\, dw_\pm^2 \pm \frac{2\, dw_\pm\, dz}{1-z}\,, \tag{II.9}$$

where $w_\pm = t \pm \ln(z/(1-z))$. In these coordinates the curvature $\tilde{R} = 1 - z$.

The field equation follows from the action

$$W[\varphi] = -\frac{1}{2}\int dX^2\, g^{1/2}\, \left[(\nabla\varphi)^2 + (m^2 + \xi R)\,\varphi^2\right]\,. \tag{II.10}$$

and has the form

$$D_{m,\xi}\varphi = 0\,, \qquad D_{m,\xi} = \Box - m^2 - \xi R\,. \tag{II.11}$$

In these relations ξ is a parameter of non-minimal coupling. The stress-energy tensor, $T_{\mu\nu}$, for the field reads

$$\begin{aligned} T_{\mu\nu} \equiv \frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{\mu\nu}} &= (1-2\xi)\,\varphi_{,\mu}\,\varphi_{,\nu} \\ &+ \left(2\xi - \frac{1}{2}\right) g_{\mu\nu}\, g^{\alpha\beta}\,\varphi_{,\alpha}\,\varphi_{,\beta} \\ &-2\xi\varphi\,(\varphi_{;\mu\nu} - g_{\mu\nu}\,\Box\varphi) \\ &+\xi\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R\right)\varphi^2 - \frac{1}{2}m^2\varphi^2 g_{\mu\nu}. \end{aligned} \tag{II.12}$$

The field equation (II.11) can be written as

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_*^2} - U\right]\varphi = 0\,, \tag{II.13}$$

where $\mu = m r_0$,

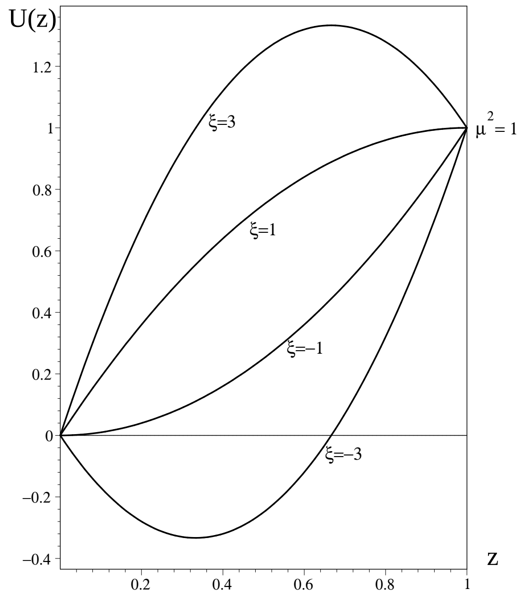

FIG. 1. Potential $U(z)$ for $\mu^2 = 1$ as a function of z. Horizon is at $z = 0$.

$$\begin{aligned} U &= f(x)\,(\mu^2 + \xi R) \\ &= \frac{\mu^2 + \xi}{1 + \exp(-x_*)} - \frac{\xi}{(1 + \exp(-x_*))^2}\,, \end{aligned} \tag{II.14}$$

and x_* is "tortoise" coordinate

$$x_* \equiv \int_1^x \frac{dx}{f(x)} = \ln(e^x - 1)\,. \tag{II.15}$$

Solutions of this equation can be constructed from monochromatic waves $\varphi \sim \mathrm{e}^{-i\omega t} R(x|\omega)$ where "radial" modes $R(x|\omega)$ obey the equation

$$\left[\frac{\partial^2}{\partial x_*^2} + \omega^2 - U\right] R(x|\omega) = 0\,. \tag{II.16}$$

Properties of solutions depend on the form of the potential U. Using the relation

$$z = 1 - \exp(-x) = 1/(1 + \exp(-x_*)\,, \tag{II.17}$$

one gets

$$U(z) = z((\mu^2 + \xi) - \xi z)\,. \tag{II.18}$$

Thus

$$U'(z) = (\mu^2 + \xi) - 2\xi z\,, \qquad U''(z) = -2\xi\,. \tag{II.19}$$

Function $U(z)$ has maximum (minimum) inside the interval $z \in (0, 1)$ when $\xi > \mu^2$ ($\xi < -\mu^2$). For $|\xi| < \mu^2$ function $U(z)$ is a monotonic function in the same interval. Qualitatively its behavior is presented in Figure 1.

Let us consider the case $\xi < -\mu^2$. Denote $\eta = -\xi$. The potential $U(z)$ is negative in the domain $z \in (0, z_1)$ where $z_1 = 1 - \mu^2/\eta$. For a fixed value of mass μ the depth of the potential well depends on ξ. The larger $|\xi|$ for negative ξ the deeper the well is. Hence there always exist negative We consider these states and condition of their formation in Section 4.

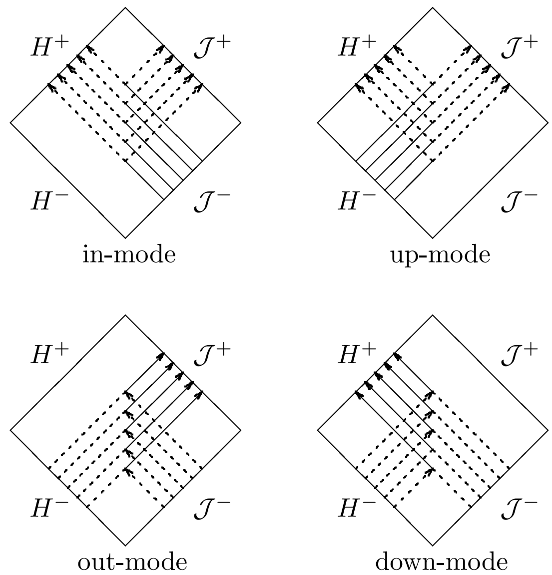

FIG. 2. *IN, UP, OUT* and *DOWN*-modes .

III. BLACK HOLE RADIATION

A. Scattering modes

Let us discuss now the scattering problem for this potential. As usual, we introduce the sets of solutions of equation (II.16) $R_{\rm up}$ and $R_{\rm in}$ which are specified by their asymptotic behavior as follows

$$R_{\rm in}(x_*|\omega) \sim \frac{1}{\sqrt{4\pi}} \begin{cases} \omega^{-1/2}\, T_\omega\, e^{-i\omega x_*}\ , & x_* \to -\infty\ , \\ \varpi^{-1/2}\left[R_\omega\, e^{i\varpi x_*} + e^{-i\varpi x_*}\right]\ , & x_* \to +\infty\ , \end{cases} \tag{III.1}$$

$$R_{\rm up}(x_*|\omega) \sim \frac{1}{\sqrt{4\pi}} \begin{cases} \omega^{-1/2}\ \left[e^{i\omega x_*} + r_\omega\, e^{-i\omega x_*}\right]\ , & x_* \to -\infty\ , \\ \varpi^{-1/2}\, t_\omega\, e^{+i\varpi x_*}\ , & x_* \to +\infty\ , \end{cases} \tag{III.2}$$

where

$$\varpi = \sqrt{\omega^2 - \mu^2}\,. \tag{III.3}$$

We also denote

$$R_{\rm out}(x|\omega) = (R_{\rm in}(x|\omega))^*\ , \qquad R_{\rm down}(x|\omega) = (R_{\rm up}(x|\omega))^*\,. \tag{III.4}$$

Modes $R_{\rm in}$ and $R_{\rm out}$ are defined for $\omega > \mu$, while modes $R_{\rm up}$ and $R_{\rm down}$ are defined for $\omega > 0$. It should be emphasized that for $0 < \omega < \mu$ we define $\varpi = i\sqrt{\mu^2 - \omega^2}$, so that in this domain of frequencies modes $R_{\rm up}$ are exponentially decreasing at large values of x_*. In the absence of bound states the modes $R_{\rm in}$ and $R_{\rm up}$ and their complex conjugated form a complete set (basis). We consider this case first and discuss aspects of the problem connected with the existence of bound states later.

Evaluating Wronskians for these solutions at $x_* = \pm\infty$, one can show that for $\omega > \mu$

$$|R_\omega|^2 + |T_\omega|^2 = 1\,, \tag{III.5}$$

$$|r_\omega|^2 + |t_\omega|^2 = 1\,, \tag{III.6}$$

and

$$T_\omega = t_\omega\,. \tag{III.7}$$

For $0 < \omega < \mu$ a wave emitted from the horizon does not reach an infinity and is totally reflected. For these waves $|R_\omega| = 1$.

Only two of four solutions, $R_{\rm up}$, $R_{\rm in}$, $R_{\rm down}$, and $R_{\rm out}$, are linearly independent. By using asymptotics of these solution one can easily obtain the following relations

$$R_{\rm in}(x|\omega) = T_\omega\, R_{\rm down}(x|\omega) + R_\omega\, R_{\rm out}(x|\omega)\,. \tag{III.8}$$

$$R_{\rm up}(x|\omega) = r_\omega\, R_{\rm down}(x|\omega) + t_\omega\, R_{\rm out}(x|\omega)\,, \tag{III.9}$$

Other similar relations can be obtained by applying complex conjugation to (III.8) and (III.9).

We demonstrate now that a general solution of the field equation (II.16) can be obtained in terms of hypergeometric functions. For this purpose we write $R(x|\omega)$ in the form

$$R(x|\omega) = q_\omega(z)\, u(z|\omega)\,, \tag{III.10}$$

where

$$q_\omega(z) = z^{-i\omega}\,(1-z)^{-i\varpi}\,. \tag{III.11}$$

One can check that the function $u(z|\omega)$ obeys the following hypergeometric equation

$$z(1-z)\,\frac{d^2u}{dz^2} + [c-(a+b+1)z]\,\frac{du}{dz} - ab\,u = 0\,, \tag{III.12}$$

where

$$a = \alpha - \beta\,, \qquad b = \alpha + \beta\,, \qquad c = 1 - 2i\omega\,, \tag{III.13}$$

$$\alpha = \frac{1}{2} - i(\omega + \varpi)\,, \qquad \beta = \sqrt{\frac{1}{4} - \xi}\,. \tag{III.14}$$

Consider first the following two linearly independent solutions

$$u_1 = F(a,b;c;z)\,, \qquad u_2 = F(a,b;\tilde{c};1-z)\,, \tag{III.15}$$

where

$$\tilde{c} = 1 - 2i\varpi\,. \tag{III.16}$$

Here and later we use the same notations u_i for special solutions of the hypergeometric equation as in the section 2.9 of the book [19]. Solutions $R_{\rm up}$ and $R_{\rm in}$ which have have asymptotics (III.1) and (III.2) can be written as

$$R_{\rm in}(x|\omega) = \frac{T_\omega}{\sqrt{4\pi\omega}}\, q_\omega u_1 \equiv \frac{T_\omega}{\sqrt{4\pi\omega}}\, q_\omega(z)\, F(a,b;c;z)\,, \tag{III.17}$$

$$R_{\rm up}(x|\omega) = \frac{t_\omega}{\sqrt{4\pi\varpi}}\, q_\omega u_2 \equiv \frac{t_\omega}{\sqrt{4\pi\varpi}}\, q_\omega(z)\, F(a,b;\tilde{c};1-z)\,. \tag{III.18}$$

We use also two other different solutions of the hypergeometric equation denoted by u_5 and u_6 in [19] (section 2.9)

$$u_5 = z^{1-c}\,(1-z)^{c-a-b}\, F(1-a,1-b;2-c;z)\,, \tag{III.19}$$

$$u_6 = z^{1-c}\,(1-z)^{c-a-b}\, F(1-a,1-b;c+1-a-b;1-z)\,. \tag{III.20}$$

Notice now that

$$a^* = \begin{cases} 1-b\,, & \text{for real } \beta\,, \\ 1-a, & \text{for imaginary } \beta\,, \end{cases} \quad , \qquad b^* = \begin{cases} 1-a\,, & \text{for real } \beta\,, \\ 1-b, & \text{for imaginary } \beta\,. \end{cases} \tag{III.21}$$

Using these relations and symmetry of the hypergeometric function with respect to its first two arguments we get

$$u_5 = z^{2i\omega}\,(1-z)^{2i\varpi}\, F(a^*,b^*;c^*;z)\,, \tag{III.22}$$

$$u_6 = z^{2i\omega}\,(1-z)^{2i\varpi}\, F(a^*,b^*;\tilde{c}^*;1-z)\,. \tag{III.23}$$

Thus we have

$$R_{\rm out} = \frac{T^*_\omega}{\sqrt{4\pi\omega}}\; q_\omega\, u_5\,, \qquad R_{\rm down} = \frac{t^*_\omega}{\sqrt{4\pi\varpi}}\; q_\omega\, u_6\,. \tag{III.24}$$

B. Transition and reflection coefficients

To determine transition and reflection coefficients it is sufficient to use Kummer relations between u_i. In particular one has (see equation (2.9.41) of [19])

$$u_5 = A\, u_2 + B\, u_6\,, \tag{III.25}$$

where

$$A = \frac{\Gamma(2-c)\,\Gamma(c-a-b)}{\Gamma(1-a)\,\Gamma(1-b)}\,, \qquad B = \frac{\Gamma(2-c)\,\Gamma(a+b-c)}{\Gamma(a+1-c)\,\Gamma(b+1-c)}\,. \tag{III.26}$$

By comparing relation (III.25) with (III.9) one gets

$$T_\omega = \sqrt{\frac{\omega}{\varpi}}\,\frac{1}{A^*}\,, \qquad r_\omega = -\frac{t_\omega}{t^*_\omega}\,\frac{B}{A}\,. \tag{III.27}$$

Coefficients R_ω and t_ω can be obtained in a similar manner.

After simple transformations the coefficient T_ω can be presented as

$$T_\omega = \sqrt{\frac{\omega}{\varpi}}\,\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)\,\Gamma(\tilde{c}-1)}\,. \tag{III.28}$$

By using relations

$$\Gamma(\zeta)\,\Gamma(1-\zeta) = \frac{\pi}{\sin(\pi\zeta)}\,, \qquad \Gamma(\zeta)\,\Gamma(-\zeta) = -\frac{\pi}{\zeta\sin(\pi\zeta)}\,, \tag{III.29}$$

we can write $|T_\omega|^2$ in the form

$$|T_\omega|^2 = \frac{2\sinh(2\pi\omega)\,\sinh(2\pi\varpi)}{\cosh[2\pi(\omega+\varpi)] + \cos(2\pi\beta)}\,. \tag{III.30}$$

This relation is valid both for real and imaginary β. It is periodic in β with a period 1.

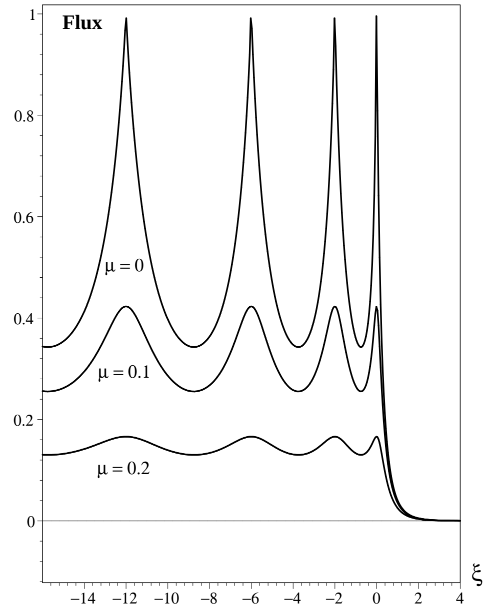

FIG. 3. The **Flux** $= 192\pi r_0^2 \times \frac{dE}{dt}$ as a function of ξ (for $\mu = 0$, $\mu = 0.1$, and $\mu = 0.2$).

C. Radiation spectrum and energy flux

The number density of particles radiated by the black hole to infinity in the range of frequencies $(\omega, \omega + d\omega)$ is given by Hawking expression

$$\frac{dn(\omega)}{d\omega} = \frac{|T_\omega|^2}{\exp(4\pi\omega) - 1} = \frac{\exp(-2\pi\omega)\sinh(2\pi\varpi)}{\cosh[2\pi(\omega + \varpi)] + \cos(2\pi\beta)} . \tag{III.31}$$

The corresponding energy density flux is (see Figure 3)

$$\frac{dE}{dt} = \frac{1}{2\pi r_0^2} \int_\mu^\infty \frac{d\omega\, \omega\, \exp(-2\pi\omega)\sinh(2\pi\varpi)}{\cosh[2\pi(\omega + \varpi)] + \cos(2\pi\beta)} . \tag{III.32}$$

(We restored the dimensionality in this relation.) For $\mu = 0$ this integral can be calculated exactly

$$\begin{aligned} \left.\frac{dE}{dt}\right|_{\mu=0} &= \frac{1}{32\pi^3 r_0^2} \frac{e^{2\pi i\beta}\,\mathrm{dilog}(1 + e^{-2\pi i\beta}) - \mathrm{dilog}(1 + e^{2\pi i\beta})}{e^{2\pi i\beta} - 1} \\ &\equiv \frac{1}{32\pi^3 r_0^2} \frac{e^{\pi\sqrt{4\xi-1}}\,\mathrm{dilog}(1 + e^{-\pi\sqrt{4\xi-1}}) - \mathrm{dilog}(1 + e^{\pi\sqrt{4\xi-1}})}{e^{\pi\sqrt{4\xi-1}} - 1} . \end{aligned} \tag{III.33}$$

Here $\mathrm{dilog}(z)$ is the dilogarithm function

$$\mathrm{dilog}(z) = -\int_1^z \frac{\ln t}{t - 1}\, dt . \tag{III.34}$$

Let us emphasize that in the above calculations of the energy flux we excluded a contribution of possible bound states, which are discussed in the next section.

IV. BOUND STATE AND BLACK HOLE INSTABILITY

A. Bound states

Solving the field equation we assumed that ω is real. Besides these wave-like solutions the system can have modes with time dependence $\sim \exp(\pm\Omega t)$. For this modes

$$q_{\Omega}(z) = z^{\Omega}\,(1-z)^{\tilde{\Omega}}\,, \tag{IV.1}$$

$$\tilde{\Omega} = \sqrt{\Omega^2+\mu^2}\,, \tag{IV.2}$$

and a corresponding "radial" function is

$$R(x|\Omega) = q_{\Omega}(z)\,u(z|\Omega)\,. \tag{IV.3}$$

If $u(z|\Omega)$ is finite both at $z=0$ and $z=1$, the radial function for $\Re(\Omega)>0$ has decreasing asymptotics both at the horizon and infinity. We shall call these states bound states.

As usual we denote by u_1 a solution which is finite at $z=0$ and by u_2 a solution which is finite at $z=1$. These solutions are given by relation (III.15) with

$$a = \frac{1}{2}+\Omega+\tilde{\Omega}-\sqrt{\frac{1}{4}-\xi}\,,\qquad b = \frac{1}{2}+\Omega+\tilde{\Omega}+\sqrt{\frac{1}{4}-\xi}\,, \tag{IV.4}$$

$$c = 1+2\Omega\,,\qquad \tilde{c} = 1+2\tilde{\Omega}\,. \tag{IV.5}$$

A bound state exists if for a given Ω solutions u_1 and u_2 are linearly dependent. It follows from Kummer relations (see (2.9.25) and (2.9.26) of [19]) that the solutions are linearly dependent if and only if either a or b is a non-positive integer number. For $\Re(\Omega)>0$ this condition can be satisfied only if both $\beta \equiv \sqrt{1/4-\xi}$ and Ω are real. A condition which determines Ω is

$$\Omega_n+\tilde{\Omega}_n = B-n\,,\qquad B\equiv \beta-\frac{1}{2}\,, \tag{IV.6}$$

where $n\ge 0$. Bound states are possible only if the non-minimal coupling constant ξ is negative. The number of bound states is defined by the integer part of the quantity $[B-\mu+1]$. Solving (IV.6) we get

$$\Omega_n = \frac{1}{2}\left[(B-n)-\frac{\mu^2}{B-n}\right]\,. \tag{IV.7}$$

When the parameter ξ is negative and decreases further, the number of bound states increases. A new bound state appears when $B-\mu$ reaches a new integer number value. In particular, transition from a pure continuous spectrum to a spectrum with a single bound state occurs when $B=\mu$ or

$$-\,\xi = \mu(\mu+1)\,. \tag{IV.8}$$

The condition $a=-n$ which determines a bound states implies that the series in the definition of the hypergeometric function is truncated and in fact the corresponding solution $u=f_n$ is a polynomial of power n

$$f_n(z) = \sum_{k=0}^{n}\frac{(-n)_k\,(b_n)_k\,z^k}{(c_n)_k\,k!}\,, \tag{IV.9}$$

where $(p)_0=1$ and $(p)_k = p(p+1)...(p+k-1)$ for $k=1,2,3,\dots$. In particular, we have

$$f_0 = 1\,,\qquad f_1 = 1-\frac{b_n}{c_n}\,z\,,\qquad f_2 = 1-2\,\frac{b_n}{c_n}\,z+\frac{b_n\,(b_n+1)}{c_n\,(c_n+1)}\,z^2\,. \tag{IV.10}$$

Since for $\Omega_n\ge 0$ both $b_n\ge 1$ and $c_n\ge 1$, the functions f_n and their first n derivatives do not vanish at $z=0$.

B. Energy density and energy fluxes for bound states

Let us calculate now the stress-energy tensor for a bound state mode. For our purposes it is convenient to make calculations in the advanced time coordinates ($v \equiv w_+, z$), (II.9). For simplicity we consider the first bound state $n = 0$. For this state

$$\varphi(v,z) = e^{\Omega_0 v}\,(1-z)^B\,. \tag{IV.11}$$

We use here the property $\Omega_0 + \tilde{\Omega}_0 = B$. Calculations made by using GRTensor give

$$\begin{aligned}
T_{\mu\nu} &= \varphi^2(v,z)\,t_{\mu\nu}\,, && \text{(IV.12)}\\
t_{vv} &= \frac{1}{4B^2}\left[B^4\left((6B+3+4B^2)+(-4+10B+16B^2)z+2(4B+3)(2B+1)z^2\right)\right.\\
&\quad \left.-2B^2\left((5B+2+4B^2)+(B+1)(8B-1)z\right)\mu^2+(2B+1)^2\mu^4\right]\,, && \text{(IV.13)}\\
t_{vz} &= \frac{1}{2(1-z)}\left[B^2\,\left((4B^2-2+2B)+(10B+8B^2+3)\,z\right)+(1-4B-4B^2)\,\mu^2\right]\,, && \text{(IV.14)}\\
t_{zz} &= \frac{B^2(2B+1)^2}{(z-1)^2}\,. && \text{(IV.15)}
\end{aligned}$$

We used (IV.7) to express Ω_0 in terms of B and μ.

The obtained result allows one to show that the energy density flux through the event horizon, T_{vv} is

$$T_{vv}|_H = \frac{B^2-\mu^2}{4B^2}\,\exp((B^2-\mu^2)v/B)\left[B^2\,(4B^2+6B+3)-\mu^2\,(2B+1)^2\right]\,. \tag{IV.16}$$

In the presence of a bound state (when $B > \mu$) the energy density flux $T_{vv}|_H$ is positive and grows exponentially with the advanced time parameter v. This behavior reflects instability of the quantum system in the presence of bound states. Quantization of scalar fields in the presence of imaginary frequency modes was discussed by Kang [20] (see also references therein).

V. GREEN FUNCTION

A. Massive field case

By making Wick's rotation $t \to i\tau$ in the metric (II.4) one gets the Euclidean metric of the form

$$ds_E^2 = f\,d\tau^2 + \frac{dx^2}{f}\,. \tag{V.1}$$

The condition of regularity at $x = 1$ requires that the coordinate τ has the period 4π.

The Euclidean Green function $G(X,X')$ which contains a complete information about the quantum field is a solution of the equation

$$\left[\Box_E - \mu^2 - \xi\,\tilde{R}\right]\,G(X,X') = -\delta(X,X')\,. \tag{V.2}$$

The Green function is assumed to be regular at the Euclidean horizon $r = r_0$ and to vanish at infinity, $r \to \infty$. Equation (V.2) in the metric (II.1) has the following form

$$\left[f^{-1}\frac{\partial^2}{\partial\tau^2} + \frac{\partial}{\partial x}\left(f\frac{\partial}{\partial x}\right) - \mu^2 - \xi\tilde{R}\right]G(X,X') = -\delta(\tau-\tau')\,\delta(x-x')\,, \tag{V.3}$$

where $\mu = mr_0$. This equation allows a separation of variables, so that we can write

$$G(X,X') = \frac{1}{4\pi}\left[\mathcal{G}_0(x,x') + 2\sum_{n=1}^{\infty}\cos(\frac{n}{2}(\tau-\tau'))\,\mathcal{G}_n(x,x')\right] \tag{V.4}$$

where $\mathcal{G}_n$ are "radial" Green functions which are solutions of the following 1-dimensional problem

$$\left[\frac{d}{dx}(f\frac{d}{dx}) - \mu^2 - \frac{n^2}{4f} - \xi\tilde{R}\right]\mathcal{G}_n(x,x') = -\delta(x-x')\,. \tag{V.5}$$

By multiplying this equation by $f(x)$ and introducing new "tortoise" coordinate

$$x_* \equiv \int_1^x \frac{dx}{f(x)} = \ln(e^x - 1)\,, \tag{V.6}$$

one can rewrite (V.6) as

$$\left[\frac{d^2}{dx_*^2} - U_n\right]\mathcal{G}_n(x,x') = -\delta(x_* - x'_*)\,, \tag{V.7}$$

where

$$U_n = \frac{n^2}{4} + U\,, \tag{V.8}$$

and is U given by (II.14). The function U_n has the asymptotic values $n^2/4$ and $\mu^2 + n^2/4$ at $x_* = -\infty$ and $x_* = \infty$, respectively. We denote by $R_n^>$ a solution of the homogeneous equation which has a decreasing asymptotic $\exp(-\sqrt{\mu^2+n^2/4}\,x_*)$ at $x_* = \infty$, and we denote by $R_n^<$ a solution which has a non-increasing asymptotic $\exp(nx_*/2)$ at $x_* = -\infty$.

To obtain a solution R_n of the homogeneous equation we denote

$$R_n(x) = z^{n/2}\,(1-z)^{\mu_n}\,u(z)\,, \tag{V.9}$$

where

$$z = 1 - e^{-x}\,, \qquad \mu_n = \sqrt{\mu^2 + \frac{n^2}{4}}\,. \tag{V.10}$$

The function $u(z)$ obeys the hypergeometric equation (III.12) with

$$a = \mu_n + \frac{n+1}{2} - \beta\,, \qquad b = \mu_n + \frac{n+1}{2} + \beta\,, \qquad c = n+1\,, \qquad \beta = \sqrt{\frac{1}{4} - \xi}\,. \tag{V.11}$$

Two linear independent solutions of this equations are

$$R_n^<(x) = z^{n/2}\,(1-z)^{\mu_n}\,F(a,b;n+1;z)\,, \tag{V.12}$$

and

$$R_n^>(x) = z^{n/2}\,(1-z)^{\mu_n}\,F(a,b;\tilde{c};1-z)\,, \qquad \tilde{c} = 2\mu_n + 1\,. \tag{V.13}$$

The functions $R_n^<$ are regular at the horizon $x = 0$ ($z = 0$) and functions $R_n^>$ are decreasing at infinity $x = \infty$ ($z = 1$).

To construct the Green function we need to calculate the Wronskian

$$W[R_n^>, R_n^<] = \frac{dR_n^<(x)}{dx_*}R_n^>(x) - \frac{dR_n^>(x)}{dx_*}R_n^<(x)\,. \tag{V.14}$$

The value of the Wronskian for the equation (V.7) does not depend on the point. For this reason it is sufficient to calculate its value at any point. It is convenient to perform calculations at $x = 1$. For this purpose we first transform the solution $R_n^>$ into the form which has the same argument as $R_n^<$. Notice that $\tilde{c} = a + b - n$. Using relation (15.3.12) of [22] we get

$$F(a, b; a+b-n; 1-z) = \frac{\Gamma(n)\Gamma(\tilde{c})}{\Gamma(a)\Gamma(b)} z^{-n} \sum_{k=0}^{n-1} \frac{(a-n)_k\,(b-n)_k}{k!(1-n)_k} z^k$$

$$-\frac{(-1)^n\Gamma(\tilde{c})}{\Gamma(a-n)\Gamma(b-n)} \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{k!(k+n)_k} z^k \left[\ln z - \psi(k+1) - \psi(k+n+1) + \psi(a+k) + \psi(b+k)\right] . \tag{V.15}$$

Here

$$(p)_k = \frac{\Gamma(p+k)}{\Gamma(p)} , \tag{V.16}$$

and $\psi(\zeta)$ is the digamma function $\psi(\zeta) = \Gamma'(\zeta)/\Gamma(\zeta)$. The relation (V.15) is valid for $|\arg(z)| < \pi$ and $|z| < 1$. In the vicinity of $z = 0$ the solutions have the following form

$$R_n^<(x) = z^{n/2}\, F_n^<(z) , \tag{V.17}$$

$$R_n^>(x) = z^{-n/2}\, F_n^>(z) , \qquad \text{for } n \geq 1 ,$$

$$R_0^>(x) = \ln z\, F_0^>(z) , \qquad \text{for } n = 0 , \tag{V.18}$$

where

$$F_n^<(0) = 1 , \qquad F_{n\geq 1}^>(0) = \frac{\Gamma(n)\,\Gamma(\tilde{c})}{\Gamma(a)\,\Gamma(b)} , \qquad F_0^>(0) = -\frac{\Gamma(\tilde{c})}{\Gamma(a)\,\Gamma(b)} . \tag{V.19}$$

Using these relations we get

$$W[R_n^>, R_n^<] \equiv \mathcal{W}_n = \frac{\Gamma(n+1)\,\Gamma\,(2\mu_n + 1)}{\Gamma(a)\,\Gamma(b)} . \tag{V.20}$$

Combining the above results we obtain the following expression for the 'radial' Green function

$$\mathcal{G}_n(x, x') = \frac{1}{\mathcal{W}_n} R_n^>(x^>) R_n^<(x^<) , \tag{V.21}$$

where

$$x^> = \max(x, x') , \qquad x^< = \min(x, x') . \tag{V.22}$$

B. Massless field case

In case when $\mu = 0$ the Green function can be obtained in an explicit form. For massless fields the modes $R_n^{\gtrless}$ simplify a little and can be expressed in terms of associated Legendre functions $P_{-\beta}^n$

$$R_n^<(x) = [z(1-z)]^{n/2}\, F\left(n + \frac{1}{2} + \beta, n + \frac{1}{2} - \beta; n+1; z\right) \tag{V.23}$$

$$= (-1)^n \frac{\Gamma(n+1)\Gamma\left(-n + \frac{1}{2} - \beta\right)}{\Gamma\left(n + \frac{1}{2} - \beta\right)}\, P_{-\frac{1}{2}-\beta}^{\ n}(1-2z) , \tag{V.24}$$

$$R_n^>(x) = [z(1-z)]^{n/2}\, F\left(n + \frac{1}{2} + \beta, n + \frac{1}{2} - \beta; n+1; 1-z\right) \tag{V.25}$$

$$= (-1)^n \frac{\Gamma(n+1)\Gamma\left(-n + \frac{1}{2} - \beta\right)}{\Gamma\left(n + \frac{1}{2} - \beta\right)}\, P_{-\frac{1}{2}-\beta}^{\ n}(-1+2z) , \tag{V.26}$$

where $z = 1 - \mathrm{e}^{-x} = (1 + \mathrm{e}^{x_*})^{-1}$. The Wronskian reads

$$\mathcal{W}_n = \frac{\Gamma(n+1)^2}{\Gamma\left(n + \frac{1}{2} + \beta\right)\Gamma\left(n + \frac{1}{2} - \beta\right)} , \tag{V.27}$$

Hence, the Green functions $\mathcal{G}_n$ become

$$\mathcal{G}_n^(x, x') = \frac{(-1)^n \pi}{\cos(\pi\beta)} \frac{\Gamma\left(-n + \frac{1}{2} - \beta\right)}{\Gamma\left(n + \frac{1}{2} - \beta\right)} P^{\ n}_{-\frac{1}{2}-\beta}(-1 + 2z^<) \ P^{\ n}_{-\frac{1}{2}-\beta}(1 - 2z^>) , \tag{V.28}$$

Substitution of these formulas into the expression (V.4) leads to the series

$$\begin{aligned} G(X, X') = \frac{1}{4\cos(\pi\beta)} \Big[& P_{-\frac{1}{2}-\beta}(-1 + 2z^<) \ P_{-\frac{1}{2}-\beta}(1 - 2z^>) \\ +2\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{n(\tau - \tau')}{2}\right) & \frac{\Gamma\left(-n + \frac{1}{2} - \beta\right)}{\Gamma\left(n + \frac{1}{2} - \beta\right)} P^{\ n}_{-\frac{1}{2}-\beta}(-1 + 2z^<) \ P^{\ n}_{-\frac{1}{2}-\beta}(1 - 2z^>)\Big] . \end{aligned} \tag{V.29}$$

Fortunately this summation can be performed completely [23] and we obtain a very simple explicit answer for the Green function of massless nonminimal fields

$$G(X, X') = \frac{1}{4\cos(\pi\beta)} \ P_{-\frac{1}{2}-\beta}(-\lambda) , \tag{V.30}$$

where

$$\lambda = (1 - 2z)(1 - 2z') + 4\sqrt{zz'(1 - z)(1 - z')} \cos\left(\frac{\tau - \tau'}{2}\right) . \tag{V.31}$$

VI. CALCULATION OF $\langle\varphi^2\rangle^{\mathrm{REN}}$

A. $\langle\varphi^2\rangle^{\mathrm{ren}}$ on the horizon

Before studying $\langle\varphi^2(x)\rangle^{\mathrm{ren}}$ in the black hole exterior we consider a special case when the point X is located on the horizon, $X = X_0$. In this case $\langle\varphi^2\rangle^{\mathrm{ren}}$ can be calculated exactly. It occurs because the Euclidean horizon is a fixed point of the Killing vector field. For this reason the Green function $G(X, X_0)$ does not depend on $\tau - \tau'$ and only $n = 0$ contributes to this quantity in the series (V.4). The functions $R_n^<$ are normalized so that $R_0^<(x = 0) = 1$ (see (V.17) and (V.19)). Using (V.15) and keeping only divergent and finite at the horizon terms in $R_0^>$, we get

$$G(X, X_0) \approx -\frac{1}{4\pi}\left[\ln(1 - e^{-x}) + 2\gamma + \psi\left(\mu + \frac{1}{2} + \beta\right) + \psi\left(\mu + \frac{1}{2} - \beta\right)\right] , \tag{VI.1}$$

where $\gamma = -\psi(1)$ is Euler's constant.

To get the renormalized value of $\langle\varphi^2\rangle$ we need first to subtract the divergence from (VI.1) and then to take the coincidence limit $X \to X_0$

$$\langle\varphi^2(x = 0)\rangle^{\mathrm{ren}} = \lim_{X \to X_0} \left(G(X, X_0) - G^{\mathrm{div}}(X, X_0)\right) . \tag{VI.2}$$

The divergent part of the Green function is (see Appendix)

$$G^{\mathrm{div}}(X, X') = -\frac{1}{4\pi}\left[\ln\left(\frac{1}{2}\mu^2\sigma(X, X')\right) + 2\gamma\right] . \tag{VI.3}$$

For $X' = X_0$, $\sigma(X, X_0) = l^2/2$ where the proper distance from the horizon l is

$$l = x + 2\ln\left(1 + \sqrt{1 - \exp(-x)}\right) . \tag{VI.4}$$

After expansion of the right-hand sides of Eqs.(VI.1-VI.3) in powers of x and cancellation of the divergences one gets

$$\langle\varphi^2(x=0)\rangle^{\text{ren}} = \frac{1}{4\pi}\left[2\ln\mu - \psi\left(\mu + \frac{1}{2} + \sqrt{\frac{1}{4} - \xi}\right) - \psi\left(\mu + \frac{1}{2} - \sqrt{\frac{1}{4} - \xi}\right)\right] . \tag{VI.5}$$

This answer is in agreement with the result of the paper [21], where $\langle\varphi^2(x=0)\rangle^{\text{ren}}$ has been calculated for a particular value of $\xi = 1/4$.

B. $\langle\varphi^2\rangle^{\text{ren}}$ outside the horizon

To obtain $\langle\varphi^2\rangle^{\text{ren}}$ outside the horizon we use the series representation for the Green function (V.4). Using mode decomposition of the divergent part of the Green function G^{div} derived in the Appendix we get

$$\langle\varphi^2(x)\rangle^{\text{ren}} = \frac{1}{4\pi}\left[\mathcal{G}_0(x,x) + \ln(\mu^2 f) + 2\gamma + 2\sum_{n=1}^{\infty}\left(\mathcal{G}_n(x,x) - \frac{1}{n}\right)\right] , \tag{VI.6}$$

where $\mathcal{G}_n$ are given by (V.21).

For given parameters μ and ξ, plots of $\langle\varphi^2(x)\rangle^{\text{ren}}$ can be constructed by using numerical calculations. Before discussing a behavior of $\langle\varphi^2(x)\rangle^{\text{ren}}$ we first check that the series in (VI.6) converges. To study large n asymptotic of the terms of the series we use WKB approximation. Notice that solutions $R_n^<$ and $R_n^>$ can be written in the following form

$$R_n(x) = \frac{1}{\sqrt{W_n(x)}}\, e^{\pm\int_{x_1}^{x} W_n(x')\,dx'} , \tag{VI.7}$$

where $W(x)$ is a solution of the equation

$$W_n^2 = U_n - \frac{1}{2}\frac{W_n''}{W_n} + \frac{3}{4}\frac{(W_n')^2}{W_n^2} , \tag{VI.8}$$

and $()' = d()/dx_*$. In relation (VI.7) signs $+$ and $-$ stand for solutions, $R_n^<$ and $R_n^>$, respectively. The Wronskian for these solutions, (V.14), is 1, and hence partial radial Green functions in the coincidence limit can be presented in the form

$$\mathcal{G}_n(x,x) = \frac{1}{2W_n(x)} . \tag{VI.9}$$

It should be emphasized that this result is exact provided $W_n(x)$ is an exact solution of the equation (VI.8). To obtain the large n asymptotic of $\mathcal{G}_n(x,x)$ it is sufficient to use an approximate solution

$$(W_n^{(1)})^2 = U_n = \frac{n^2}{4} + f(x)\,(\mu^2 + \xi\tilde{R}) . \tag{VI.10}$$

Iterations show that the omitted terms in equation (VI.8) are of higher order in $1/n$ expansion. In this approximation we get

$$\mathcal{G}_n^{(1)}(x,x) = \frac{1}{n} - \frac{2}{n^3}\, f(x)\,(\mu^2 + \xi\tilde{R}) . \tag{VI.11}$$

This result demonstrates that the subtraction of $1/n$ term makes the series in (VI.6) convergent. Moreover, one can use the WKB result to improve the convergence. By adding and subtracting $\mathcal{G}_n^{(1)}(x,x)$ and using the definition of the Riemann zeta function

$$\zeta(s) = \sum_{1}^{\infty} n^{-s}\,, \tag{VI.12}$$

one can present $\langle\varphi^2(x)\rangle^{\text{ren}}$ in the form

$$\langle\varphi^2(x)\rangle^{\text{ren}} = \frac{1}{4\pi}\left[\mathcal{G}_0(x,x) + \ln(\mu^2 f) + 2\gamma - 4\zeta(3) f(x)\,(\mu^2 + \xi\tilde{R}) + \Delta\mathcal{G}(x,x)\right]\,, \tag{VI.13}$$

$$\Delta\mathcal{G}(x,x) = 2\sum_{n=1}^{\infty}\left(\mathcal{G}_n(x,x) - \mathcal{G}_n^{(1)}(x,x)\right)\,. \tag{VI.14}$$

The terms of the series decrease as n^{-5}. The convergence can still be improved by using higher in n^{-1} corrections in the WKB expansion for $\mathcal{G}_n(x,x)$.

C. Massless case

Because in the massless case we know the Green function explicitly, calculation of the $\langle\varphi^2\rangle^{\text{ren}}$ is greatly simplified. It is convenient to rewrite the answer Eq.(V.30) in the form

$$G(X,X') = \frac{1}{4\cos(\pi\beta)}\;F\left(\frac{1}{2}+\beta, \frac{1}{2}-\beta;\; 1;\; \frac{1+\lambda}{2}\right) \tag{VI.15}$$

For small point splitting $\lambda = 1 - 2\epsilon$, where $\epsilon \to 0$. Let us consider the separation in z-direction ($\tau - \tau' = 0$) . Then the proper distance between the points reads

$$\begin{aligned}\rho - \rho' = l(z,z') &= 2\ln(1+\sqrt{z}) - \ln(1-z) - 2\ln(1+\sqrt{z'}) + \ln(1-z') \\ &= \frac{1}{(1-z)\sqrt{z}}\;|z-z'| + O\left((z-z')^2\right)\end{aligned} \tag{VI.16}$$

and

$$\begin{aligned}\epsilon &= z + z' - 2zz' - 2\sqrt{zz'(1-z)(1-z')} \\ &= \frac{1}{4z(1-z)}(z-z')^2 + O\left((z-z')^3\right) = \frac{1-z}{4}\;l^2 + O(l^3)\;.\end{aligned} \tag{VI.17}$$

In this limit the degenerate hypergeometric function is

$$F\left(\frac{1}{2}+\beta, \frac{1}{2}-\beta; 1; 1-\epsilon\right) = \frac{\cos(\pi\beta)}{\pi}\left[-\ln(\epsilon) - 2\gamma - \psi\left(\frac{1}{2}+\beta\right) - \psi\left(\frac{1}{2}-\beta\right)\right] + O(\epsilon^2)\;. \tag{VI.18}$$

Thus

$$G(X,X') = -\frac{1}{4\pi}\;\left[\ln(1-z) + \ln\left(\frac{l^2}{4}\right) + 2\gamma + \psi\left(\frac{1}{2}+\beta\right) + \psi\left(\frac{1}{2}-\beta\right)\right] + O(l^3)\;. \tag{VI.19}$$

Taking into account Eq.(VI.3)

$$G(X,X')^{\text{div}} = -\frac{1}{4\pi}\;\left[\ln\left(\frac{\mu^2 l^2}{4}\right) + 2\gamma\right] \tag{VI.20}$$

we obtain

$$\langle\varphi^2(X)\rangle^{\rm ren} = \lim_{X'\to X}\left(G(X,X') - G^{\rm div}(X,X')\right) \tag{VI.21}$$

$$= \frac{1}{4\pi}\left[-\ln(1-z) + \ln(\mu^2) - \psi\left(\frac{1}{2}+\beta\right) - \psi\left(\frac{1}{2}-\beta\right)\right] \tag{VI.22}$$

$$= \frac{1}{4\pi}\left[x + \ln(\mu^2) - \psi\left(\frac{1}{2}+\beta\right) - \psi\left(\frac{1}{2}-\beta\right)\right]\,. \tag{VI.23}$$

On the horizon (at $x = 0$) this result, evidently, reproduces the massless limit $\mu \to 0$ of the Eq.(VI.5), the mass μ playing the role of the infrared cut-off parameter.

VII. CONCLUSION

We studied quantum effects for quantum nonminimal scalar field in a two-dimensional black hole spacetime. We demonstrated that for a string motivated black hole metric [8,9] the field equation allows exact analytical solutions in terms of hypergeometric functions. Using these solutions we obtained an explicit expression for greybody factors and calculated Hawking radiation. We also demonstrated that for negative values of nonminimal coupling constant ξ the field besides usual scattering modes can have bound states. The bound states for the scalar field of mass m near the black hole with the gravitational radius r_0 are present when $-\xi \le \mu(\mu+1)$, where $\mu = mr_0$. These bound states lead to instability. As a result of this the negative energy density in the region in the black hole exterior grows exponentially. This effect is accompanied by exponentially growing positive energy fluxes through the black hole horizon and to infinity. It should be emphasized that this kind of instability occurs for any theory with negative ξ for solutions describing evaporating black holes, since the gravitational radius $r_0 \to 0$ and there is a moment of time when the parameter μ meets the condition of formation of a bound state. This result may indicate that 2D theories of the nonminimal scalar field is inconsistent for negative ξ.

The obtained explicit expressions for Green functions and $\langle\varphi^2\rangle^{\rm ren}$ may be used to test an accuracy of different analytical approximations developed for study of vacuum polarization effects in the spacetime with the black hole.

Acknowledgments: This work was partly supported by the Natural Sciences and Engineering Research Council of Canada. The authors are grateful to the Killam Trust for its financial support.

APPENDIX A: DIVERGENCIES

The divergent part of the Green function in two dimensions can be obtained by integrating Schwinger-DeWitt expansion of the heat kernel,

$$K^{\rm div}(X,X'|s) = \frac{D^{\frac{1}{2}}(X,X')}{4\pi s}\exp\left\{-\mu^2 s - \frac{\sigma(X,X')}{2s}\right\}[1+\ldots]\,, \tag{A.1}$$

Here σ is the half of the square of geodesic distance and D is Van Fleak-Morette determinant. One has

$$G^{\rm div}(X,X') = \frac{D^{\frac{1}{2}}(X,X')}{4\pi}\int_0^\infty ds\ \frac{1}{s}\exp\left\{-\mu^2 s - \frac{\sigma(X,X')}{2s}\right\}$$

$$
\begin{aligned}
&= \frac{D^{\frac{1}{2}}(X,X')}{2\pi} K_0(\mu\sqrt{2\sigma(X,X')})| \\
&= \frac{D^{\frac{1}{2}}(X,X')}{4\pi}\left[-\ln\left(\frac{\mu^2\sigma(X,X')}{2}\right) - 2\gamma\right].
\end{aligned}
$$

For splitting points in τ direction in the metric (II.1) one has ($\Delta\tau = \tau - \tau'$)

$$
\begin{aligned}
2\sigma(X,X') &= f(x)\Delta\tau^2 + O(\Delta\tau^4) = 8f(x)\left[1 - \cos\left(\frac{\Delta\tau}{2}\right)\right] + O(\Delta\tau^4)\,, \\
D^{\frac{1}{2}}(X,X') &= 1 + O(\Delta\tau^2)\,.
\end{aligned}
$$

Thus

$$
G^{\mathrm{div}}(\tau,x;\tau',x) = -\frac{1}{4\pi}\left[\ln\left(2\left[1-\cos\left(\frac{\tau-\tau'}{2}\right)\right]\right) + \ln\left(\mu^2 f\right) + 2\gamma\right]. \tag{A.2}
$$

Using the relation

$$
\ln\left(2\left[1-\cos\left(\frac{\tau-\tau'}{2}\right)\right]\right) = \sum_{k=1}^{\infty}\left[-\frac{2}{k}\cos\left(k\frac{\tau-\tau'}{2}\right)\right]
$$

we get

$$
G^{\mathrm{div}}(\tau,x;\tau',x) = \frac{1}{4\pi}\left[-\ln\left(\mu^2 f\right) - 2\gamma + 2\sum_{n=1}^{\infty}\cos\left(\frac{n}{2}(\tau-\tau')\right)\times\frac{1}{n}\right]
$$

From this one can easily obtain for the Fourier components of the UV divergent part of Green function

$$
\begin{aligned}
\mathcal{G}_0^{\mathrm{div}}(x,x) &= -\ln\left(\mu^2 f\right) - 2\gamma\,, \\
\mathcal{G}_n^{\mathrm{div}}(x,x) &= \frac{1}{n}\,.
\end{aligned}
$$

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